**SYDNEY BOYS HIGH SCHOOL**
**MOORE PARK, SURRY HILLS**

**2015**

**HIGHER SCHOOL CERTIFICATE**
**TRIAL PAPER**

# Mathematics Extension 1

*General Instructions*

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Leave your answers in the simplest exact form, unless otherwise stated.
- Start each **NEW** question in a separate answer booklet.

**Total Marks – 70**

**Section I** Pages 2–4

**10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

**Section II** Pages 6–11

**60 marks**

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Examiners: *R. Elliot & J. Chen*

**This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.**

**Section I**

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** The roots of $3x^3 - 2x^2 + x - 1 = 0$ are $\alpha$, $\beta$ and $\gamma$.
What is the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ ?

(A) $-\frac{1}{9}$

(B) $-\frac{2}{9}$

(C) 1

(D) $\frac{2}{9}$

**2** What is the minimum value of $\sqrt{7}\sin x - 3\cos x$ ?

(A) –2

(B) –4

(B) –16

(D) $\sqrt{7} - 3$

**3** What is the domain and range of $y = \sin^{-1}\left(\frac{2x}{5}\right)$?

(A) Domain: $-1 \le x \le 1$; Range: $-\pi \le y \le \pi$

(B) Domain: $-1 \le x \le 1$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) Domain: $-\frac{5}{2} \le x \le \frac{5}{2}$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(D) Domain: $-\frac{5}{2} \le x \le \frac{5}{2}$; Range: $-\pi \le y \le \pi$

**4** Evaluate $\lim_{x \to 0} \frac{\sin 3x}{2x}$.

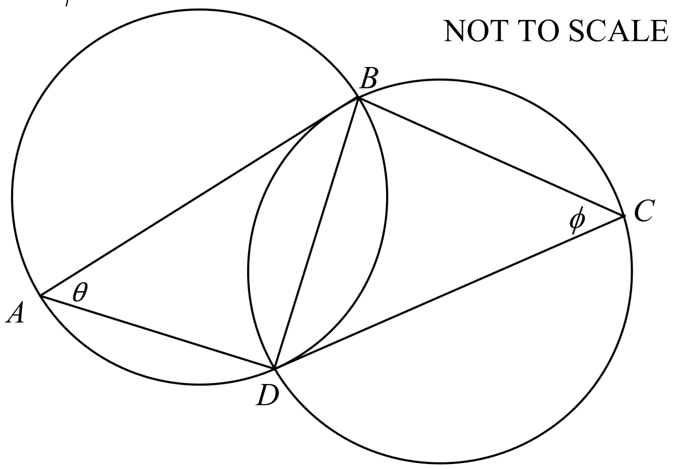
(A) 0

(B) $\frac{2}{3}$

(C) 1

(D) $\frac{3}{2}$

**5** In the diagram below, $AB$ is a tangent to the circle $BCD$.
Also, $CD$ is a tangent to the circle $ABD$.
$\angle BAD = \theta$ and $\angle BCD = \phi$.




Which of the following is a true statement?

(A) $\Delta ABD \equiv \Delta BDC$

(B) $ABCD$ is a cyclic quadrilateral

(C) $\Delta ABD \parallel\!| \Delta BDC$

(D) $AB \parallel CD$

**6** A particle moves in simple harmonic motion so that its velocity, $v$, is given by

$$v^2 = 6 - x - x^2 .$$

Between which two points does it oscillate?

(A) $x = 6$ and $x = 3$

(B) $x = -2$ and $x = 3$

(C) $x = 1$ and $x = 2$

(D) $x = 2$ and $x = -3$

**7** Which of the following is an expression for $\int \cos^3 x \sin x \, dx$ ?

(A) $-\cos^4 x + c$

(B) $-\frac{1}{4}\cos^4 x + c$

(C) $\cos^4 x + c$

(D) $\frac{1}{4}\cos^4 x + c$

**8** Which of the following is the correct expression for the inverse of $f(x) = e^{1-2x}$ ?

(A) $f^{-1}(x) = -2e^{1-2x}$

(B) $f^{-1}(x) = -\frac{1}{2}e^{1-2x}$

(C) $f^{-1}(x) = -\frac{1}{2}\log_e(1-2x)$

(D) $f^{-1}(x) = \frac{1}{2}(1-\log_e x)$

**9** Three Mathematics study guides, four Mathematics textbooks and five exercise books are randomly placed along a bookshelf. What is the probability that the Mathematics textbooks are all next to each other?

(A) $\frac{4!}{12!}$

(B) $\frac{9!}{12!}$

(C) $\frac{4!3!5!}{12!}$

(D) $\frac{4!9!}{12!}$

**10** A particle moves on the $x$-axis with velocity $v$ m/s, such that $v^2 = 16x - x^2$.
Which of the following is the particle's maximum speed and the position of where this maximum speed occurs?

(A) Maximum speed = 16 m/s at $x = 0$

(B) Maximum speed = 8 m/s at $x = -8$

(C) Maximum speed = –8 m/s at $x = 8$

(D) Maximum speed = 8 m/s at $x = 8$

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**Section II**

**60 marks**
**Attempt Questions 11–14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 Marks) Start a NEW Writing Booklet

(a) Differentiate $\sin^{-1}(\log_e x)$. **1**

(b) Find $\int \frac{1}{\sqrt{4-9x^2}}\,dx$. **1**

(c) (i) Simplify $\sin(A+B)+\sin(A-B)$. **1**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{6}} \sin 3x \cos x\, dx$. **2**

(d) The point $P(6p, 3p^2)$ is a point on the parabola $x^2 = 12y$.

(i) Find the equation of the tangent at *P*. **2**

(ii) The tangent at *P* cuts the *y*-axis at *B*. **3**
The point *A* divides *PB* internally in the ratio 1 : 2.
Find the locus of the point *A* as *P* varies.

(e) A piece of meat at temperature *T*° C is placed in an oven, which has a constant temperature of *H*° C.
The rate at which the temperature of the meat warms is given by

$$\frac{dT}{dt} = -K(T-H),$$

where *t* is in minutes and for some positive constant *K*.

(i) Show that $T = H + Be^{-Kt}$, where *B* is a constant, is a solution of the differential equation above. **1**

(ii) If the meat warms from 10° C to 50° C in the oven, which has a constant temperature of 180° C, in 30 minutes, find the value of *K*. **2**

(iii) How long will it take the meat to get to a temperature of 150° C? **2**
Express your answer correct to the nearest minute.

**Question 12** (15 Marks) Start a NEW Writing Booklet

(a) (i) Solve $\cos x - \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$. **2**

(ii) Hence, or otherwise, find a general solution to $\cos x - \sqrt{3}\sin x = 1$. **1**

(b) (i) On the same set of axes sketch the graphs of $y = \cos 2x$ and $y = \dfrac{x+1}{3}$ **2**

(ii) Use the graph to determine the number of solutions to the equation **1**

$$3\cos 2x = x + 1$$

(iii) One solution of the equation $3\cos 2x = x + 1$ is close to 0.5. **3**
Use one application of Newton's Method to find another approximation, correct to 3 decimal places.

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$ **3**

(d) When $x$ cm from the origin, the acceleration of a particle moving in a straight line is given by: **3**

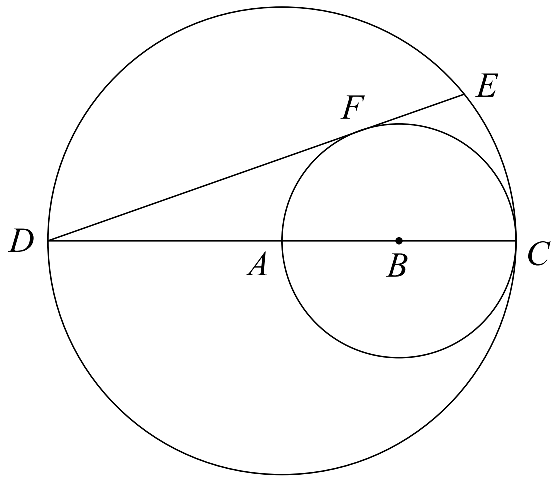
$$\frac{d^2x}{dt^2} = -\frac{5}{(x+2)^3}$$

It has an initial velocity of 2 cm/s at $x = 0$. If the velocity is $V$ cm/s, find $V$ in terms of $x$.

**Question 13** (15 Marks) Start a NEW Writing Booklet

(a) In the diagram below, $DC$ is a diameter of the larger circle centred at $A$. **4**
$AC$ is a diameter of the smaller circle centred at $B$.
$DE$ is tangent to the smaller circle at $F$ and $DC = 12$.

Copy the diagram to your answer booklet.
Determine the length of $DE$.




(b) (i) Simplify $k! + k \times k!$ **1**

(ii) Prove, by mathematical induction, that **3**

$$1\times 1! + 2\times 2! + 3\times 3! + ... + n\times n! = (n+1)! - 1$$

for all positive integers $n$.

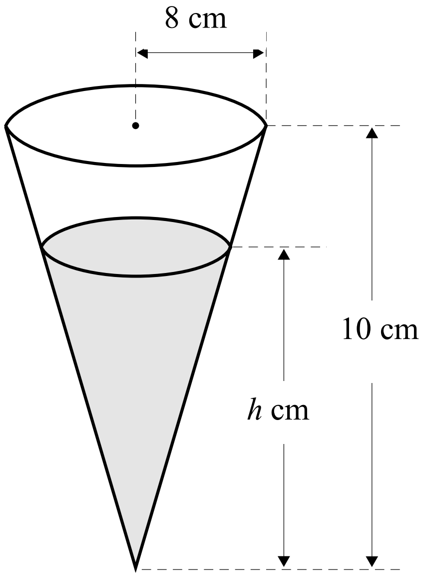
(c) (i) Using the substitution $x = 3 + 3\sin\theta$ find $\int \sqrt{x(6-x)}\, dx$ **4**

(ii) Let $R$ be the region bounded by the curve $y = \sqrt[4]{x(6-x)}$ and the $x$-axis. **3**
Find the volume of the solid of revolution generated by revolving $R$ about the $x$-axis.

**End of Question 13**

**Question 14** (15 Marks) Start a NEW Writing Booklet

(a)




The figure above shows an inverted conical cup with base radius 8 cm and height 10 cm.

Some water is poured into the cup at a constant rate of $\frac{2\pi}{5}$ $cm^3$ per minute.

Let the depth of the water be $h$ cm at time $t$ minutes.

Find the rate of change in the area of the water surface when $h = 4$ **3**

(b) A particle is projected horizontally at 30 $ms^{-1}$ from the top of a 100 m high wall. Assume that acceleration due to gravity is 10 $ms^{-2}$ and that there is no air resistance.

The flight path of the particle is given by:

$$x = 30t,\ y = 100 - 5t^2 \text{ (Do NOT prove this)}$$

where $t$ is the time in seconds after take-off.

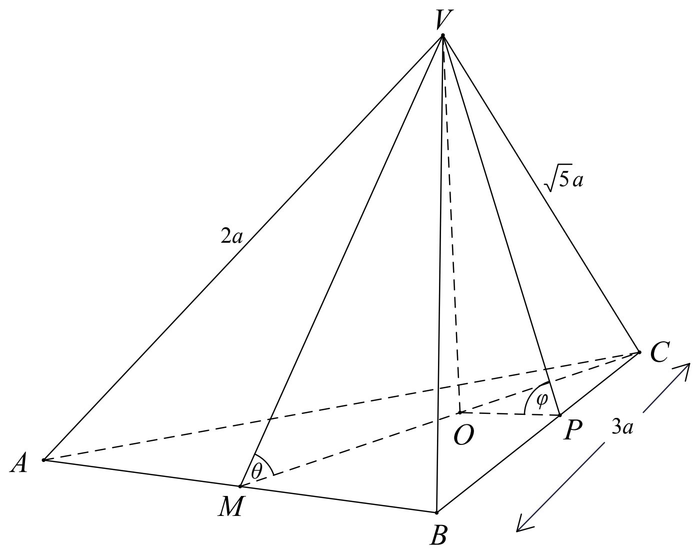
(i) Find the time taken for the particle to reach the ground. **1**

(ii) Find the angle and speed at which the particle strikes the ground. **2**

**Question 14 continues on page 11**

Question 14 (continued)

(c) The diagram below shows a tetrahedron such that $VA = VB = AB = 2a$, $CA = CB = 3a$ and $VC = \sqrt{5}a$.
$O$ is the foot of the perpendicular from $V$ to the base $ABC$.
$M$ is the midpoint of $AB$.
$P$ is a point on $BC$ such that $BP = ra$ where $0 \le r \le 3$.
$\angle VMC = \theta$ and $\angle VPO = \varphi$.




(i) By considering $\Delta VMC$, show that $\cos\theta = \dfrac{\sqrt{6}}{4}$. **3**

(ii) Hence find the exact value of $VO$. **1**

(iii) Show that $VP^2 = \dfrac{1}{3}\left(3r^2 - 8r + 12\right)a^2$ **2**

(iv) Hence show that $\sin\varphi = \sqrt{\dfrac{45}{8\left(3r^2 - 8r + 12\right)}}$ **1**

(v) Hence, or otherwise, find the maximum value of $\varphi$ as $r$ varies. **2**

**End of paper**

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# Mathematics Extension 1

# Sample Solutions

| Question | Teacher |
|---|---|
| Q11 | **RB** |
| Q12 | **BK** |
| Q13 | **BD** |
| Q14 | **PB** |

## MC Answers

| | |
|---|---|
| Q1 | D |
| Q2 | B |
| Q3 | C |
| Q4 | D |
| Q5 | C |
| Q6 | D |
| Q7 | B |
| Q8 | D |
| Q9 | D |
| Q10 | D |

**Section I** **10 marks**

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1 The roots of $3x^3 - 2x^2 + x - 1 = 0$ are $\alpha$, $\beta$ and $\gamma$.
What is the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$?

(A) $-\frac{1}{9}$

(B) $-\frac{2}{9}$

(C) 1

(D) $\frac{2}{9}$

**ANSWER: D**

$3x^3 - 2x^2 + x - 1 = 0$

$\alpha\beta\gamma = -\frac{d}{a}$ $\qquad$ $\alpha + \beta + \gamma = -\frac{b}{a}$

$= \frac{1}{3}$ $\qquad$ $= \frac{2}{3}$

$$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma)$$
$$= \frac{1}{3} \times \frac{2}{3}$$
$$= \frac{2}{9}$$

2 What is the minimum value of $\sqrt{7}\sin x - 3\cos x$?

(A) –2

(B) –4

(B) –16

(D) $\sqrt{7} - 3$

**ANSWER: B**

$\sqrt{7}\sin x - 3\cos x$

$r = \sqrt{\left(\sqrt{7}\right)^2 + 3^2}$

$= \sqrt{7 + 9}$

$= 4$

Let $\sqrt{7}\sin x - 3\cos x = r\sin(\theta - \alpha)$

$= 4\sin(\theta - \alpha)$

No matter what the value of $\alpha$

$-1 \le \sin(x - \alpha) \le 1$

$-4 \le 4\sin(x - \alpha) \le 4$

Therefore the minimum value is $x = -4$.

**3** What is the domain and range of $y = \sin^{-1}\left(\frac{2x}{5}\right)$?

(A) Domain: $-1 \le x \le 1$; Range: $-\pi \le y \le \pi$

(B) Domain: $-1 \le x \le 1$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) Domain: $-\frac{5}{2} \le x \le \frac{5}{2}$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(D) Domain: $-\frac{5}{2} \le x \le \frac{5}{2}$; Range: $-\pi \le y \le \pi$

**ANSWER: C**

$$y = \sin^{-1}\left(\frac{2x}{5}\right) \Rightarrow \sin y = \left(\frac{2x}{5}\right)$$

Domain: $-1 \le \sin y \le 1$

$$-1 \le \frac{2x}{5} \le 1$$

$$\frac{-5}{2} \le x \le \frac{5}{2}$$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ as this is the range of $y = \sin^{-1} x$

**4** Evaluate $\lim_{x \to 0} \frac{\sin 3x}{2x}$.

(A) $0$

(B) $\frac{2}{3}$

(C) $1$

(D) $\frac{3}{2}$

**ANSWER: D**

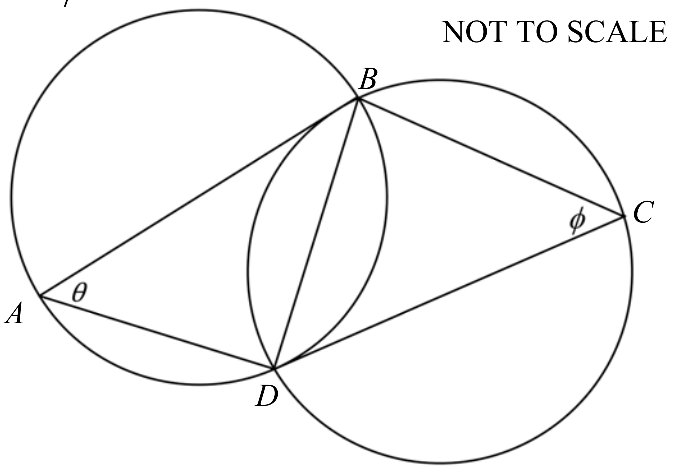
$$\lim_{x \to 0} \frac{\sin 3x}{2x} = \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{2x} \times \frac{2}{3}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{2 \sin 3x}{2 \times 3x}$$

$$= \frac{3}{2} \times 1$$

$$= \frac{3}{2}$$

**5** In the diagram below, $AB$ is a tangent to the circle $BCD$.
Also, $CD$ is a tangent to the circle $ABD$.
$\angle BAD = \theta$ and $\angle BCD = \phi$.




Which of the following is a true statement?

(A) $\Delta ABD \equiv \Delta BDC$

(B) $ABCD$ is a cyclic quadrilateral

(C) $\Delta ABD \parallel\!\mid \Delta BDC$

(D) $AB \parallel CD$

**ANSWER: C**

In $\Delta ABD$ and $\Delta DCB$ :
$\angle DCB = \angle DBA$ (angle in the alternate segment)
ie $\angle DCB = \angle ABD$
$\angle BAD = \angle BDC$ (angle in the alternate segment)
ie $\angle BAD = \angle CDB$
$BD$ is but not respective to angles.
Therefore $\Delta ABD \not\equiv \Delta DCB$

Hence, triangles are equiangular they are similar and $\Delta ABD \parallel\!\mid \Delta DCB$

**6** A particle moves in simple harmonic motion so that its velocity, $v$, is given by

$$v^2 = 6 - x - x^2.$$

Between which two points does it oscillate?

(A) $x = 6$ and $x = 3$

(B) $x = -2$ and $x = 3$

(C) $x = 1$ and $x = 2$

(D) $x = 2$ and $x = -3$

**ANSWER: D**

$v^2 = 6 - x - x^2$
For the particle to reach its oscillation points $v = 0$.

$$v^2 = 6 - x - x^2$$
$$0 = 6 - x - x^2$$
$$0 = (3 + x)(2 - x)$$
$$\therefore x = -3 \text{ and } 2$$

**7** Which of the following is an expression for $\int \cos^3 x \sin x \, dx$?

(A) $-\cos^4 x + c$

(B) $-\frac{1}{4}\cos^4 x + c$

(C) $\cos^4 x + c$

(D) $\frac{1}{4}\cos^4 x + c$

**ANSWER: B**

$\int \cos^3 x \sin x \, dx$, testing solutions:

$$\frac{d}{dx}\left(\cos^4 x\right) = 4\cos^3 x \times -\sin x$$

$$\frac{d}{dx} \; -\frac{1}{4}\left(\cos^4 x\right) = \cos^3 x \sin x$$

$$-\frac{1}{4}\cos^4 x = \int \cos^3 x \sin x \, dx$$

**8** Which of the following is the correct expression for the inverse of $f(x) = e^{1-2x}$?

(A) $f^{-1}(x) = -2e^{1-2x}$

(B) $f^{-1}(x) = -\frac{1}{2}e^{1-2x}$

(C) $f^{-1}(x) = -\frac{1}{2}\log_e(1-2x)$

(D) $f^{-1}(x) = \frac{1}{2}(1-\log_e x)$

**ANSWER: D**

Let $y = e^{1-2x}$

$$y = e^{1-2x}$$

$$\ln y = 1 - 2x$$

$$2x = 1 - \ln y$$

$$x = \frac{1}{2}\left(1 - \ln y\right)$$

$$\therefore f^{-1}(x) = \frac{1}{2}\left(1 - \ln y\right)$$

**9** Three Mathematics study guides, four Mathematics textbooks and five exercise books are randomly placed along a bookshelf. What is the probability that the Mathematics textbooks are all next to each other?

(A) $\frac{4!}{12!}$

(B) $\frac{9!}{12!}$

(C) $\frac{4!3!5!}{12!}$

(D) $\frac{4!9!}{12!}$

**ANSWER: D**

Since there are 9 elements counting the textbooks as 1 element, hence these can be arranged in 9! ways. Also the textbooks can be arranged in 4! ways.

As there are 12 separate elements, the divisor for population can be counted in 12! ways.

Therefore, the probability is $\frac{9!4!}{12!}$

**10** A particle moves on the $x$-axis with velocity $v$ m/s, such that $v^2 = 16x - x^2$. Which of the following is the particle's maximum speed and the position of where this maximum speed occurs?

(A) Maximum speed = 16 m/s at $x = 0$

(B) Maximum speed = 8 m/s at $x = -8$

(C) Maximum speed = –8 m/s at $x = 8$

(D) Maximum speed = 8 m/s at $x = 8$

**ANSWER: D**

$$v^2 = 16x - x^2$$

$$\frac{1}{2}v^2 = 8x - \frac{x^2}{2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8 - \frac{2x}{2}$$

$$= 8 - x$$

$$\ddot{x} = 8 - x$$

When $\ddot{x} = 0$ the speed is the greatest so,

$\ddot{x} = 8 - x$

$0 = 8 - x$

$x = 8$

At $x = 8$,

$$v^2 = 16x - x^2$$

$$= 16(8) - (8)^2$$

$$= 64$$

$$v = \pm 8$$

As v, velocity can take positive and negative values, but the speed can only be positive, the maximum speed is $8$ m/s.

# (11) 3 unit Trial 2015 Sydney Boys.

$\frac{\ }{15}$

(a) let $y = \sin^{-1}(\ln x)$

$$y' = \frac{1}{\sqrt{1-(\ln x)^2}} \times \frac{1}{x}$$

$$= \frac{1}{x\sqrt{1-(\ln x)^2}} \quad (1)$$

generally well answered but some students forgot the $\frac{1}{x}$. Others thought $(\ln x)^2 = 2\ln x$ or $\ln x^2$ No!

(b) $$\int \frac{1\ dx}{\sqrt{4-9x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C \quad (1)$$

generally well done but some students left off the $\frac{1}{3}$.

(C) (i) $\sin(A+B) + \sin(A-B)$

$\sin A\cos B + \cos A \sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B \quad (1)$

this part, very well answered by 95% of students.

(ii) $$\int_0^{\frac{\pi}{6}} \sin 3x \cos x \ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 2\sin 3x\cos x \ dx$$

$$= \frac{1}{2}\int \left[\sin 4x + \sin 2x\right] dx$$

$$= -\frac{1}{2}\cdot\frac{1}{4}\int 4\sin 4x \ dx + -\frac{1}{2}\cdot\frac{1}{2}\int 2\sin 2x \ dx$$

$$= -\frac{1}{8}\cos 4x\Big]_0^{\frac{\pi}{6}} - \frac{1}{4}\cos 2x\Big]_0^{\frac{\pi}{6}}$$

11 (d) (ii) $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

ratio formula often messed up!

$P(\overset{x_1}{6p}, \overset{y_1}{3p^2})$

$B(\overset{x_2}{0}, \overset{y_2}{-3p^2})$

$m:n = 1:2$

$A = \left(\frac{1\times 0 + 12p}{3}, \frac{-3p^2+6p^2}{3}\right)$

$A\ (4p, p^2)$ (1)

So

$x = 4p \Rightarrow p = \frac{x}{4}$

$y = p^2$

So $y = \frac{x^2}{16}$ $\Rightarrow$ $x^2 = 16y$ is the locus. (1)

Many students found 'A' but could not find locus equation.

---

(e) (i) $T = H + Be^{-kt}$ $\Rightarrow Be^{-kt} = T - H$

$\frac{dT}{dt} = 0 - Bke^{-kt}$

$= -k(T-H)$ (1)

(ii) $H = 180^\circ$

When $t = 0$, $T = 10^\circ$

$10 = 180 + Be^0$

$10 - 180 = B$

$B = -170$

So $T = 180 - 170e^{-kt}$

data $t = 30$

$T = 50$

Well answered. Using the method shown. Some boys wanted to integrate the $\frac{dT}{dt}$ and got into a lot of bother.

So $50 = 180 - 170e^{-30k}$

$-130 = -170e^{-30k}$

$\left(\frac{13}{17}\right) = e^{-30k}$

$\ln\left(\frac{13}{17}\right) = \ln e^{-30k}$

$= -30k$

$k = \frac{\ln\left(\frac{13}{17}\right)}{-30} = \cdot 008942$

Luckily k either as exact or approx will give the correct exact or approx. final answer.

So $T = 180 - 170e^{-0.008942t}$ (2)

(iii) $150 = 180 - 170e^{-0.008942t}$

$-30 = -170e^{-0.008942t}$

$\left(\frac{3}{17}\right) = e^{-0.008942t}$

$\ln\left(\frac{3}{17}\right) = \ln e^{-0.008942t}$

$= -0.008942t$

$t = \frac{\ln\left(\frac{3}{17}\right)}{-0.008942} \doteq 194$ mins (2)

3hrs 14 mins

Well answered.

$$\left(-\frac{1}{8}\cos\frac{4\pi}{6} + \frac{1}{8}\cos 0\right) - \left(\frac{1}{4}\cos\frac{2\pi}{6} - \frac{1}{4}\cos 0\right)$$

$$= -\frac{1}{8}\cos\frac{2\pi}{3} + \frac{1}{8} - \left(\frac{1}{4}\cos\frac{\pi}{3} - \frac{1}{4}\right)$$

$$= -\frac{1}{8}\cos 120^\circ + \frac{1}{8} - \frac{1}{4}\cos 60^\circ + \frac{1}{4}$$

$$= -\frac{1}{8} \times \frac{-1}{2} + \frac{1}{8} - \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}$$

$$= \frac{1}{16} + \frac{1}{8} - \frac{1}{8} + \frac{1}{4} = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$ //

(0.3125)

Many students did not use hint from (c)(i) or used it badly. The $\frac{1}{2}$ out the front was there in most answers. Those that made the connection that (i) can help in (ii), most students worked out the answer.

(2)

(d) $P(6p, 3p^2)$ $x^2 = 12y$

(i) $x^2 = 12y$

$y = \frac{x^2}{12}$

$y' = \frac{2x}{12} = \frac{x}{6}$

at $x = 6p$ $m = \frac{6p}{6} = p$

$(y - 3p^2) = p(x - 6p)$

well answered.

$y - 3p^2 = px - 6p^2$

$y = px - 3p^2$ (2)

(ii) Cuts y axis at B.

$x = 0$ $y = -3p^2$

$B(0, -3p^2)$. (1)

easy to find

$P(6p, 3p^2)$.

12. (a) (i) $\cos x - \sqrt{3}\sin x = 1 \qquad 0 \le x \le 2\pi$

$\cos x - \sqrt{3}\sin x = R\cos(x+\alpha)$

$\cos x - \sqrt{3}\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\Rightarrow \quad R = \sqrt{1+3} = 2$

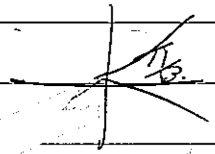
Then $2\cos\alpha = 1$ and $2\sin\alpha = \sqrt{3}$.

$\cos\alpha = \frac{1}{2} \qquad \sin\alpha = \frac{\sqrt{3}}{2}$.

$\therefore \ \underline{\alpha = \frac{\pi}{3}}$ ✓

Then $2\cos\left(x+\frac{\pi}{3}\right) = 1$

$\cos\left(x+\frac{\pi}{3}\right) = \frac{1}{2}$




(2)

$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3},$ ✓

$\underline{x = 0, \frac{4\pi}{3}, 2\pi}$.

(ii) General Soln: $x = 2n\pi$ or $2n\pi \pm \frac{4\pi}{3}$. ✓

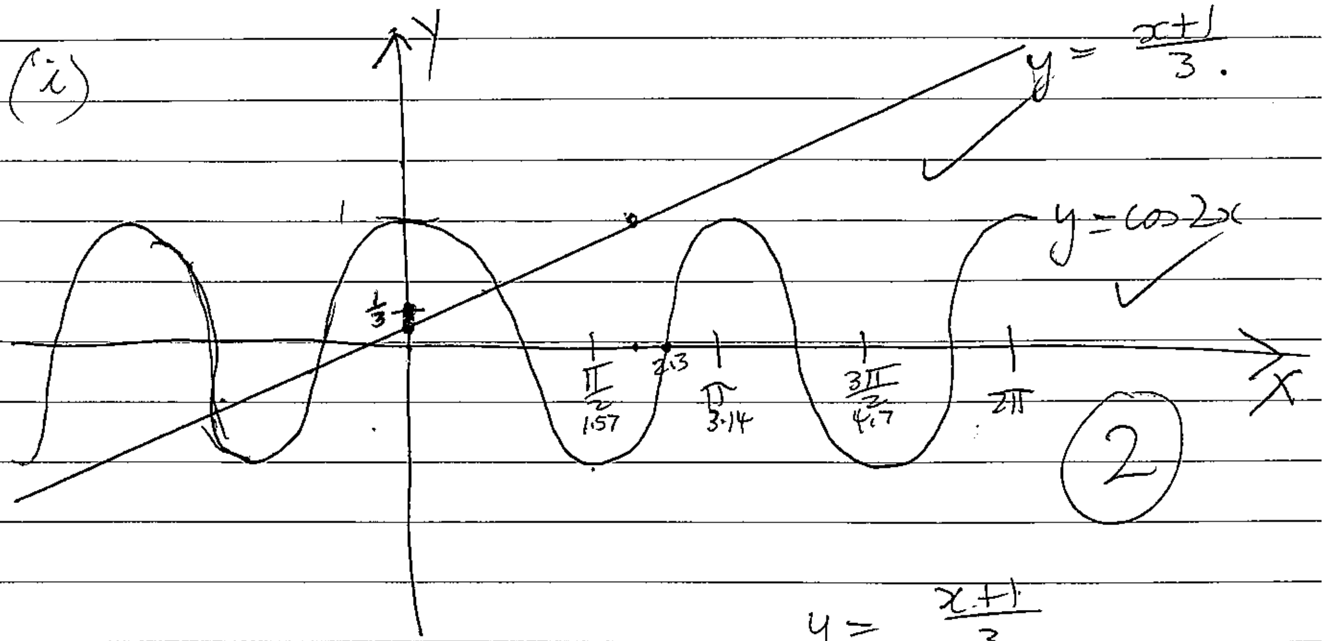
(1)

or $x = -\frac{\pi}{3} + 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$

or $x = -\frac{2\pi}{3} + 2k\pi$ or $2k\pi$ $k \in \mathbb{Z}$

(i) Most students realised they needed the auxiliary angle method. Common errors included:
-evaluating tan incorrectly and having 1/sqrt(3)
-not finding all the solutions in the given domain.
(ii) Some had the incorrect formula.

(b) (i)




(2)

$y = \frac{x+1}{3}$

| x | 0 | 2 |
|---|---|---|
| y | $\frac{1}{3}$ | 1 |

(ii) 3 solutions ✓ (1)

(iii) $f(x) = 3\cos 2x - x - 1$

Let $f(0.5) =$ $1.6209 - 0.5 - 1$

$f(0.5) = 0.1209069$

Also $f'(x) = -6\sin 2x - 1$ ✓

$f'(x) = -6.0488$

Then $x_{n+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.5 - \frac{0.1209069}{-6.0488259}$ ✓ (3)

$= 0.5199884907$

$x_{n+1} = 0.520$ to 3dp ✓

An inaccurate graph resulted in the wrong number of solutions. Using Newton's Method done well on the whole. Some students did not use the given starting value and so were incorrect.

(c) $\int_0^{\pi/4} \sin^2 2x\,dx$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin^2 2x = \frac{1}{2}(1-\cos 4x)$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1-\cos 4x)\,dx$ ✓

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/4}$ ✓

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right) - (0-0)\right]$ ✓

$= \pi/8$

The most common mistake was not using the double angle formula correctly.

(3)

(d) $\ddot{x} = \frac{-5}{(x+2)^3}$

When $t=0$, $x = x$ cm.

$t=0$, $v=2$, $x=0$.

$\frac{d(\frac{1}{2}v^2)}{dx} = \frac{-5}{(x+2)^3}$

$\frac{1}{2}v^2 = -5\int \frac{1}{(x+2)^3}\,dx$

$v^2 = \frac{-10\,(x+2)^{-2}}{-2} + C$ ✓

$v^2 = \frac{5}{(x+2)^2} + C$ ✓

When $x=0, v=2 \Rightarrow 4 = \frac{5}{4} + C$

$C = 2\frac{3}{4} = \frac{11}{4}$

(d) (cont)

$$v^2 = \frac{5}{(x+2)^2} + \frac{11}{4}$$

$v > 0$ since $v$ can never be 0

Also when $t = 0$, $v = 2$ and

$\frac{d^2x}{dt^2} < 0 \Rightarrow$ decreasing speed.

$$\therefore v = \sqrt{\frac{5}{(x+2)^2} + \frac{11}{4}}$$

3

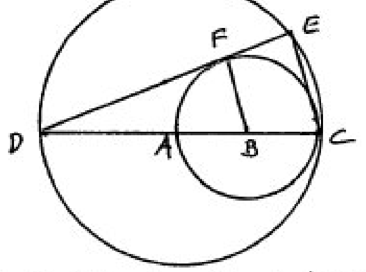
The most common error was to differentiate the given function in terms of t.

Half a mark was deducted if no statement about sign of v was included.

Solutions Q13 X1 THSC 2015

Average mark: 11.31/15

(a)




FB ⊥ DF (radius ⊥ tangent at point of contact)

$DB^2 = FB^2 + DF^2$ (Pythagoras' Theorem)

$\therefore 9^2 = 3^2 + DF^2$

$\therefore DF^2 = 72$

$\therefore DF = 6\sqrt{2}$

EC ⊥ DE (DC is a diameter $\therefore \angle DEC = 90°$, angle in a semicircle)

$\therefore \Delta DBF \parallel\!\mid \Delta DCE$ (equiangular)

as $\angle D$ is common

$\angle DFB = \angle DEC = 90°$ (as indicated above)

$\therefore \frac{DE}{DF} = \frac{DC}{CB}$ (corresponding sides in similar triangles)

$\therefore \frac{DE}{6\sqrt{2}} = \frac{12}{9}$

$\therefore DE = \frac{12 \times 6\sqrt{2}}{9}$

$= 8\sqrt{2}$ (4)

There seemed to be a reluctance to give reasons for geometrical conclusions

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | Mean |
|---|---|---|---|---|---|---|---|---|---|
| 7 | 6 | 11 | 14 | 25 | 8 | 20 | 22 | 49 | 2.70 |

(b) (i) $k! + k \times k!$

$= k!(1+k)$

$= (k+1)!$ (1)

Done well although some stopped at $k!(1+k)$

| 0 | 0.5 | 1 | Mean |
|---|---|---|---|
| 6 | 18 | 138 | 0.91 |

(ii) $S(n) \equiv 1\times 1! + 2\times 2! + \ldots + n\times n! = (n+1)! - 1$

Show $S(1)$ is true

ie $1 \times 1! = 2! - 1$

LHS = 1

RHS = 2 − 1

= 1

$\therefore S(1)$ is true

Assume $S(k)$ is true

ie $1\times 1! + 2\times 2! + \ldots + k\times k! = (k+1)! - 1$

Show $S(k+1)$ is true

i.e $1\times 1! + 2\times 2! + \ldots + k\times k! + (k+1)\times(k+1)! = (k+2)! - 1$

LHS $= (k+1)! - 1 + (k+1)\times(k+1)!$

$= (k+1)!(1 + k+1) - 1$

$= (k+1)!(k+2) - 1$

$= (k+2)! - 1$

= RHS

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true

$S(1)$ is true and, if $S(k)$ is true, $S(k+1)$ is true

$\therefore$ By the process of Mathematical Induction, $S(n)$ is true for all integral $n \geq 1$. (3)

Most students demonstrated an understanding of the process of Mathematical Induction. However, many statements were sloppy. For example, "Assume $n=k$" rather than "Assume the statement is true if $n=k$" or, having defined the statement as $S(n)$ as above, "Assume that $S(k)$ is true". Many concluding statements were also sloppy.

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | Mean |
|---|---|---|---|---|---|---|---|
| 2 | 8 | 2 | 3 | 0 | 4 | 143 | 2.77 |

(c) (i) If $x = 3 + 3\sin\theta$,

$dx = 3\cos\theta$

and $\sin\theta = \frac{x-3}{3}$.

$$\int \sqrt{x(6-x)}\,dx$$

$$= \int \sqrt{(3+3\sin\theta)(6-3-3\sin\theta)}\,.3\cos\theta\,d\theta$$

$$= \int 3\sqrt{(1-\sin^2\theta)}\,.\,3\cos\theta\,d\theta$$

$$= 9\int \cos^2\theta\,d\theta$$

NOTE: ORIGINAL INTEGRAL IS POSITIVE

$$= 9\int \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right] + c$$

$$= \frac{9}{2}\left[\sin\theta\cos\theta + \theta\right] + c$$

$$= \frac{9}{2}\left[\frac{x-3}{3}\sqrt{1-\frac{(x-3)^2}{3}} + \sin^{-1}\frac{x-3}{3}\right] + c$$

$$= \frac{9}{2}\left[\frac{x-3}{3}\,\frac{\sqrt{6x-x^2}}{3} + \sin^{-1}\frac{x-3}{3}\right] + c$$

$$= \frac{1}{2}\left[(x-3)\sqrt{x(6-x)} + 9\sin^{-1}\frac{x-3}{3}\right] + c$$

(4)

Many students found their integration challenging. Some left the integral at the form $\frac{9}{2}[\sin\theta\cos\theta + \theta] + c$, or equivalent rather than returning to an expression in terms of $x$.

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | Mean |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 11 | 4 | 20 | 15 | 25 | 24 | 55 | 2.93 |

(ii) $V = \pi\int_0^6 \left(\sqrt[4]{x(6-x)}\right)^2 dx$

$$= \pi\int_0^6 \sqrt{x(6-x)}\,dx$$

$$= \pi \times \frac{1}{2}\left[(x-3)\sqrt{x(6-x)} + 9\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^6$$

$$= \frac{\pi}{2}\left\{\left[9\sin^{-1}1\right] - \left[9\sin^{-1}(-1)\right]\right\}$$

$$= \frac{\pi}{2}\left\{9.\frac{\pi}{2} - 9\left(-\frac{\pi}{2}\right)\right\}$$

$$= \frac{9\pi^2}{2}$$

(3)

Most who progressed through (c)(i) found the appropriate volume.

| 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | Mean |
|---|---|---|---|---|---|---|---|
| 15 | 10 | 21 | 4 | 24 | 23 | 65 | 2.05 |

14(a)

$$\frac{dV}{dt} = \frac{2\pi}{5}$$ (given)

$S = \pi r^2$  $V = \frac{1}{3}\pi r^2 h.$

$\therefore \frac{dS}{dr} = 2\pi r$

& $\frac{h}{10} = \frac{r}{8}$ (SIMILARITY)

$\therefore V = \frac{5}{12}\pi r^3$

$$\frac{dV}{dr} = \frac{5}{4}\pi r^2$$

now $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}.$

$= 2\pi r \times \frac{4}{5\pi r^2} \times \frac{2\pi}{5}$

$= \frac{16\pi}{25r}$  when $h = 4$, $r = \frac{16}{5}$

$= \frac{16}{25} \times \pi \times \frac{5}{16}.$

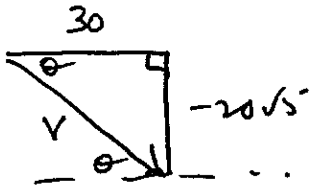
$= \frac{\pi}{5}$ cm/min

COMMENT: not particularly well done. many students treated h as a constant in the differentiation of $\frac{dV}{dr}$.

(b) (i). let $y = 0$.

$$100 - 5t^2 = 0$$
$$5t^2 = 100$$
$$t^2 = 20$$
$$t = 2\sqrt{5} \text{ secs.}$$

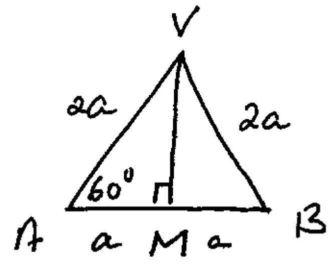
(ii) $\dot{x} = 30$ $\quad \dot{y} = -10t$
$$= -20\sqrt{5}$$



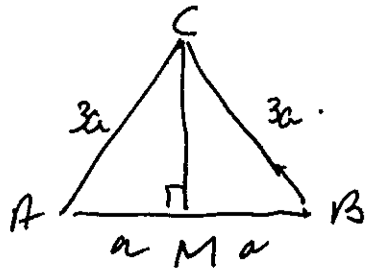

$$V^2 = 900 + 2000$$
$$= \sqrt{2900}$$
$$\boxed{V = 10\sqrt{29} \text{ m/s.}}$$
$$\boxed{\theta = \tan^{-1} \frac{2\sqrt{5}}{3}}$$

COMMENT Common error was to let $y = -100$. Generally well done.

(c).




$VM = 2a\sin 60°$
$= 2a \frac{\sqrt{3}}{2}$
$VM = a\sqrt{3}$.




$CM^2 = (3a)^2 - a^2$
$= 8a^2$
$\therefore CM = \sqrt{8a^2}$.

$$\cos\theta = CM^2 + VM^2 - VC^2$$
$$= \frac{8a^2 + 3a^2 - 5a^2}{2 \times a\sqrt{3} \times 2\sqrt{2}a}$$
$$= \frac{6a^2}{4\sqrt{6}a^2}$$
$$\therefore \boxed{\cos\theta = \frac{\sqrt{6}}{4}}$$

COMMENT: Quite well done.

(ii) $VO = VM\sin\theta$.
$= a\sqrt{3} \times \frac{\sqrt{10}}{4}$
$= \frac{a\sqrt{30}}{4}$

$\sin^2\theta = 1 - \frac{6}{16}$
$= \frac{10}{16}$
$\therefore \sin\theta = \frac{\sqrt{10}}{4}$

COMMENT:
many students unable to find $\sin\theta$.

(iii) $$\cos V\hat{B}C = \frac{(2a)^2 + (3a)^2 - (\sqrt{5}a)^2}{2 \times 2a \times 3a}$$

$$= \frac{4a^2 + 9a^2 - 5a^2}{12a^2}$$

$$= \frac{2}{3}$$

$$\therefore VP^2 = VB^2 + (ra)^2 - 2 \times 2a \times ra \times \frac{2}{3}$$

$$= 4a^2 + r^2a^2 - \frac{8ra^2}{3}$$

$$= \frac{a^2}{3}\left[12 + 3r^2 - 8r\right]$$

COMMENT Very few students were able to obtain this answer. The common error was to assume $\triangle COP \sim \triangle CMB$. Hence finding an expression for OP then using Pythagoras to obtain $VP^2$. (this was not given marks)

(iv) $$VP = a\sqrt{\frac{12+3r^2-8r}{3}}$$

$$\therefore \sin\theta = \frac{\frac{a\sqrt{30}}{4}}{a\sqrt{\frac{12+3r^2-8r}{3}}}$$

$$= \frac{\sqrt{90}}{4\sqrt{12+3r^2-8r}}$$

$$= \frac{\sqrt{45}}{\sqrt{8(12+3r^2-8r)}}$$

COMMENT most obtained a mark, allowing for previous error in VO.

(v). Best done by recognising

that $\sin\phi$ is maximised
by $3r^2-8r+12$ being a minimum
this occurs when $6r-8=0$

$$r = 4/3.$$

$$\text{i.e. } \sin\phi = \sqrt{\frac{45}{8(3r^2-8r+12)}} \quad \text{where } r=\frac{4}{3}$$

$$= \sqrt{\frac{45\times 9}{8\times 60}}$$

$$= \frac{3\sqrt{6}}{8}$$

$$\therefore \phi = \sin^{-1}\frac{3\sqrt{6}}{8}$$

COMMENT

many students were able to obtain a mark or two.

many maximised $\sin\phi$ by calculus. Few saw the easier approach.